5.5. Integriranje racionalnih funkcija

18. 12. 2020.

Fotografija: Roma Kaiuk

Osnovni trik: rastav na parcijalne razlomke prije samog integriranja

Za polinome P i $Q \neq 0$, integral

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

računamo ovako: zapišemo funkciju $\frac{P(x)}{Q(x)}$ u obliku

$$\frac{P(x)}{Q(x)} = \underbrace{p(x)}_{\text{polinom}} + \underbrace{r_1(x)} + \underbrace{r_2(x)} + \ldots + \underbrace{r_n(x)}$$

parcijalni razlomci

Osnovni trik: rastav na parcijalne razlomke prije samog integriranja

Za polinome P i $Q \neq 0$, integral

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

računamo ovako: zapišemo funkciju $\frac{P(x)}{Q(x)}$ u obliku

$$\frac{P(x)}{Q(x)} = \underbrace{p(x)}_{\text{polinom}} + \underbrace{r_1(x)} + \underbrace{r_2(x)} + \ldots + \underbrace{r_n(x)}$$

parcijalni razlomci

$$\Rightarrow \quad \int \frac{P(x)}{Q(x)} = \underbrace{\int p(x)\,\mathrm{d}x}_{\text{znamo}} + \underbrace{\int r_1(x)\,\mathrm{d}x} + \underbrace{\int r_2(x)\,\mathrm{d}x} + \ldots + \underbrace{\int r_n(x)\,\mathrm{d}x}\,.$$

naučit ćemo

Zadatak 51(a)

Izračunajte integral $I := \int \frac{x^5 + x^4 - 8}{x^3 - 4x} \, \mathrm{d}x.$

Zadatak 51(a)

Izračunajte integral $I := \int \frac{x^5 + x^4 - 8}{x^3 - 4x}\,\mathrm{d}x$.

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}$$

Zadatak 51(a)

Izračunajte integral $I := \int \frac{x^5 + x^4 - 8}{x^3 - 4x} \, \mathrm{d}x$.

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}$$

pa je

$$I = \int \left(x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2} \right) \mathrm{d}x$$

Zadatak 51(a)

Izračunajte integral $I := \int \frac{x^5 + x^4 - 8}{x^3 - 4x} \, \mathrm{d}x$.

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}$$

pa je

$$\begin{aligned} I &= \int \left(x^2 + x + 4 + \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2} \right) \mathrm{d}x \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C. \end{aligned}$$

Postoji nekoliko klasičnih trikova. . .

Klasični trik A

Za $a \in \mathbb{R} \setminus \{0\}$ i $A, B \in \mathbb{R}$,

$$\int \frac{Ax + B}{x^2 + a^2}\,\mathrm{d}x = \underbrace{\int \frac{Ax}{x^2 + a^2}\,\mathrm{d}x}_{\substack{\text{supstitucijom}\\ t=x^2+a^2}} + B \underbrace{\int \frac{\mathrm{d}x}{x^2 + a^2}}_{\text{tablični integral}}\,.$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}$.

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}.$

- $I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \left[\begin{matrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{matrix}\right]$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x.$

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}.$

- $I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \begin{bmatrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{bmatrix} = \int \frac{3}{t} \cdot \frac{\mathrm{d}t}{2}$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,dx$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,dx}_{=:I_1} - \underbrace{\int \frac{dx}{x^2+5}}_{=:I_2}.$

- $I_1 = \int \frac{3x}{x^2+5}\,dx = \begin{bmatrix} t = x^2+5 \\ dt = 2x\,dx \end{bmatrix} = \int \frac{3}{t}\cdot\frac{dt}{2}$
  $= \frac{3}{2}\ln|t| + C_1$

$$\int dx = x + C$$
$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{dx}{x} = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}.$

- $$I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \left[\begin{matrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{matrix}\right] = \int \frac{3}{t}\cdot\frac{\mathrm{d}t}{2}$$
  $$= \frac{3}{2}\ln|t| + C_1 = \frac{3}{2}\ln\left(x^2+5\right) + C_1.$$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}$.

- $$I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \begin{bmatrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{bmatrix} = \int \frac{3}{t}\cdot\frac{\mathrm{d}t}{2}$$
  $$= \frac{3}{2}\ln|t| + C_1 = \frac{3}{2}\ln\left(x^2+5\right) + C_1.$$
- $$I_2 = \int \frac{\mathrm{d}x}{x^2+5}$$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x.$

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}.$

- $I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \begin{bmatrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{bmatrix} = \int \frac{3}{t}\cdot\frac{\mathrm{d}t}{2}$
  $= \frac{3}{2}\ln|t| + C_1 = \frac{3}{2}\ln\left(x^2+5\right) + C_1.$
- $I_2 = \int \frac{\mathrm{d}x}{x^2+5} = \int \frac{\mathrm{d}x}{x^2+\left(\sqrt{5}\right)^2}$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,\mathrm{d}x$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,\mathrm{d}x}_{=:I_1} - \underbrace{\int \frac{\mathrm{d}x}{x^2+5}}_{=:I_2}$.

- $I_1 = \int \frac{3x}{x^2+5}\,\mathrm{d}x = \begin{bmatrix} t = x^2+5 \\ \mathrm{d}t = 2x\,\mathrm{d}x \end{bmatrix} = \int \frac{3}{t}\cdot\frac{\mathrm{d}t}{2}$
  $= \frac{3}{2}\ln|t| + C_1 = \frac{3}{2}\ln\left(x^2+5\right) + C_1$.
- $I_2 = \int \frac{\mathrm{d}x}{x^2+5} = \int \frac{\mathrm{d}x}{x^2+\left(\sqrt{5}\right)^2}$
  $= \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C_2$.

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(b)

Izračunajte integral $I := \int \frac{3x-1}{x^2+5}\,dx$.

Rješenje. $I \overset{\text{Trik A}}{=} \underbrace{\int \frac{3x}{x^2+5}\,dx}_{=:I_1} - \underbrace{\int \frac{dx}{x^2+5}}_{=:I_2}$.

- $I_1 = \int \frac{3x}{x^2+5}\,dx = \begin{bmatrix} t = x^2+5 \\ dt = 2x\,dx \end{bmatrix} = \int \frac{3}{t}\cdot\frac{dt}{2}$
  $= \frac{3}{2}\ln|t| + C_1 = \frac{3}{2}\ln\left(x^2+5\right) + C_1$.
- $I_2 = \int \frac{dx}{x^2+5} = \int \frac{dx}{x^2+\left(\sqrt{5}\right)^2}$
  $= \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C_2$.

$\Rightarrow \quad I = I_1 - I_2 = \frac{3}{2}\ln\left(x^2+5\right) - \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C$.

$$\int dx = x + C$$
$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{dx}{x} = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a>0,\ a\neq 1)$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a>0)$$

Klasični trik B

Za $b \in \mathbb{R} \setminus \{0\}$ i $A, B, c \in \mathbb{R}$ takve da je $D := b^2 - 4c < 0$, integral

$$\int \frac{Ax + B}{x^2 + bx + c}\,\mathrm{d}x$$

možemo izračunati ovako: zapisivanjem nazivnika $x^2 + bx + c$ u obliku

kvadrat binoma + konstanta

dobivamo

$$\begin{aligned}\int \frac{Ax + B}{x^2 + bx + c}\,\mathrm{d}x &= \int \frac{Ax + B}{\left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}}\,\mathrm{d}x \\ &= \begin{bmatrix} t = x + \dfrac{b}{2} \quad \Rightarrow \quad x = t - \dfrac{b}{2} \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix} \\ &= \int \frac{At - \frac{Ab}{2} + B}{t^2 + \frac{4c - b^2}{4}}\,\mathrm{d}t.\end{aligned}$$

Dobiveni integral možemo izračunati klasičnim trikom A!

Zadatak 51(c)

Izračunajte integral $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x$.

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(c)

Izračunajte integral $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x$.

Rješenje. $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{3x-1}{(x-2)^2+4}\,\mathrm{d}x$

$$= \begin{bmatrix} t = x-2 \;\Rightarrow\; x = t+2 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix}$$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\; a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(c)

Izračunajte integral $\displaystyle\int \frac{3x-1}{x^2-4x+8}\,dx$.

Rješenje.
$$\int \frac{3x-1}{x^2-4x+8}\,dx \overset{\text{Trik B}}{=} \int \frac{3x-1}{(x-2)^2+4}\,dx$$
$$= \begin{bmatrix} t = x-2 \;\Rightarrow\; x = t+2 \\ dt = dx \end{bmatrix}$$
$$= \frac{3(t+2)-1}{t^2+4}\,dt = \int \frac{3t+5}{t^2+4}\,dt$$

$$\int dx = x + C$$
$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{dx}{x} = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\; a \neq 1)$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(c)

Izračunajte integral $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x$.

Rješenje. $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{3x-1}{(x-2)^2+4}\,\mathrm{d}x$

$$= \begin{bmatrix} t = x-2 \;\Rightarrow\; x = t+2 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix}$$

$$= \int \frac{3(t+2)-1}{t^2+4}\,\mathrm{d}t = \int \frac{3t+5}{t^2+4}\,\mathrm{d}t$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{3t}{t^2+4}\,\mathrm{d}t}_{\substack{\text{supstitucijom } u=t^2+4 \\ \text{ili pogodimo}}} + \underbrace{\int \frac{5}{t^2+2^2}\,\mathrm{d}t}_{\text{tablični integral}}$$

$$\int \mathrm{d}x = x + C$$

$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

$$\int e^x\,\mathrm{d}x = e^x + C$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(c)

Izračunajte integral $\int \frac{3x-1}{x^2-4x+8}\,dx$.

Rješenje.

$$\int \frac{3x-1}{x^2-4x+8}\,dx \overset{\text{Trik B}}{=} \int \frac{3x-1}{(x-2)^2+4}\,dx$$

$$= \left[\begin{array}{c} t = x-2 \;\Rightarrow\; x = t+2 \\ dt = dx \end{array}\right]$$

$$= \int \frac{3(t+2)-1}{t^2+4}\,dt = \int \frac{3t+5}{t^2+4}\,dt$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{3t}{t^2+4}\,dt}_{\substack{\text{supstitucijom } u=t^2+4 \\ \text{ili pogodimo}}} + \underbrace{\int \frac{5}{t^2+2^2}\,dt}_{\text{tablični integral}}$$

$$= \frac{3}{2}\ln\left(t^2+4\right) + 5\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2} + C$$

$$\int dx = x + C$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\; a \neq 1)$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(c)

Izračunajte integral $\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x$.

Rješenje.

$$\int \frac{3x-1}{x^2-4x+8}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{3x-1}{(x-2)^2+4}\,\mathrm{d}x$$

$$= \begin{bmatrix} t = x-2 \;\Rightarrow\; x = t+2 \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix}$$

$$= \int \frac{3(t+2)-1}{t^2+4}\,\mathrm{d}t = \int \frac{3t+5}{t^2+4}\,\mathrm{d}t$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{3t}{t^2+4}\,\mathrm{d}t}_{\substack{\text{supstitucijom } u=t^2+4 \\ \text{ili pogodimo}}} + \underbrace{\int \frac{5}{t^2+2^2}\,\mathrm{d}t}_{\text{tablični integral}}$$

$$= \frac{3}{2}\ln\left(t^2+4\right) + 5\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2} + C$$

$$= \frac{3}{2}\ln\left(x^2-4x+8\right) + \frac{5}{2}\operatorname{arctg}\left(\frac{x}{2}-1\right) + C.$$

$$\int \mathrm{d}x = x + C$$

$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

$$\int e^x\,\mathrm{d}x = e^x + C$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\; a \neq 1)$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Klasični trik C*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

Klasični trik C*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2 + a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2 + a^2) - x^2}{(x^2 + a^2)^2} \, \mathrm{d}x$$

Klasični trik C*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2 + a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2 + a^2) - x^2}{(x^2 + a^2)^2} \mathrm{d}x = \frac{1}{a^2} \int \left(\frac{1}{x^2 + a^2} - \frac{x^2}{(x^2 + a^2)^2} \right) \mathrm{d}x$$

Klasični trik C*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2+a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2+a^2) - x^2}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{a^2} \int \left(\frac{1}{x^2+a^2} - \frac{x^2}{(x^2+a^2)^2} \right) \mathrm{d}x$$

$$= \frac{1}{a^2} \left(\frac{1}{a} \operatorname{arctg} \frac{x}{a} - \underbrace{\int \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}x}_{=:I_1} \right).$$

Klasični trik C*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2+a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2+a^2)^2} \overset{\text{Trik C}_1}{=} \frac{1}{a^2} \int \frac{(x^2+a^2)-x^2}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{a^2}\int \left(\frac{1}{x^2+a^2} - \frac{x^2}{(x^2+a^2)^2}\right)\mathrm{d}x$$

$$= \frac{1}{a^2}\left(\frac{1}{a}\operatorname{arctg}\frac{x}{a} - \underbrace{\int \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}x}_{=:I_1}\right).$$

Nadalje,

$$I_1 \overset{\text{Trik C}_2}{=} \int x \cdot \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x \overset{\text{p.i.}}{=} \begin{bmatrix} u = x & \mathrm{d}u = \mathrm{d}x \\ \mathrm{d}v = \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x & v = -\frac{1}{2}\cdot\frac{1}{x^2+a^2} \end{bmatrix}$$

Klasični trik C^*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2+a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2+a^2) - x^2}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{a^2} \int \left(\frac{1}{x^2+a^2} - \frac{x^2}{(x^2+a^2)^2} \right) \mathrm{d}x$$

$$= \frac{1}{a^2} \left(\frac{1}{a} \operatorname{arctg} \frac{x}{a} - \underbrace{\int \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}x}_{=:I_1} \right).$$

Nadalje,

supstitucijom $t = x^2 + a^2$

$$I_1 \overset{\text{Trik } C_2}{=} \int x \cdot \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x \overset{\text{p.i.}}{=} \begin{bmatrix} u = x & \mathrm{d}u = \mathrm{d}x \\ \mathrm{d}v = \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x & v = -\frac{1}{2} \cdot \frac{1}{x^2+a^2} \end{bmatrix}$$

Klasični trik C^*

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2+a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2+a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2+a^2)-x^2}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{1}{a^2}\int \left(\frac{1}{x^2+a^2} - \frac{x^2}{(x^2+a^2)^2} \right) \mathrm{d}x$$

$$= \frac{1}{a^2}\left(\frac{1}{a} \operatorname{arctg}\frac{x}{a} - \underbrace{\int \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}x}_{=:I_1} \right).$$

Nadalje,

supstitucijom $t = x^2 + a^2$

$$I_1 \overset{\text{Trik } C_2}{=} \int x \cdot \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x \overset{\text{p.i.}}{=} \begin{bmatrix} u = x & \mathrm{d}u = \mathrm{d}x \\ \mathrm{d}v = \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x & v = -\frac{1}{2}\cdot\frac{1}{x^2+a^2} \end{bmatrix}$$

$$= -\frac{1}{2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2}\int\frac{\mathrm{d}x}{x^2+a^2}$$

Za $a \in \mathbb{R} \setminus \{0\}$, integral

$$I := \int \frac{\mathrm{d}x}{(x^2 + a^2)^2}$$

možemo izračunati ovako:

$$I = \int \frac{\mathrm{d}x}{(x^2 + a^2)^2} \overset{\text{Trik } C_1}{=} \frac{1}{a^2} \int \frac{(x^2 + a^2) - x^2}{(x^2 + a^2)^2}\,\mathrm{d}x = \frac{1}{a^2} \int \left(\frac{1}{x^2 + a^2} - \frac{x^2}{(x^2 + a^2)^2} \right) \mathrm{d}x$$

$$= \frac{1}{a^2} \left(\frac{1}{a} \operatorname{arctg} \frac{x}{a} - \underbrace{\int \frac{x^2}{(x^2 + a^2)^2}\,\mathrm{d}x}_{=: I_1} \right).$$

Nadalje,

supstitucijom $t = x^2 + a^2$

$$I_1 \overset{\text{Trik } C_2}{=} \int x \cdot \frac{x}{(x^2 + a^2)^2}\,\mathrm{d}x \overset{\text{p.i.}}{=} \begin{bmatrix} u = x & \mathrm{d}u = \mathrm{d}x \\ \mathrm{d}v = \frac{x}{(x^2+a^2)^2}\,\mathrm{d}x & v = -\frac{1}{2} \cdot \frac{1}{x^2+a^2} \end{bmatrix}$$

$$= -\frac{1}{2} \cdot \frac{x}{x^2 + a^2} + \frac{1}{2} \int \frac{\mathrm{d}x}{x^2 + a^2} = -\frac{1}{2} \cdot \frac{x}{x^2 + a^2} + \frac{1}{2} \cdot \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C.$$

Zadatak 51(d)

Izračunajte integral $I := \int \frac{x^3 + 2x^2 - 2x + 1}{x^4 - x} \, \mathrm{d}x.$

Zadatak 51(d)

Izračunajte integral $I := \int \frac{x^3+2x^2-2x+1}{x^4-x}\,\mathrm{d}x.$

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^3+2x^2-2x+1}{x^4-x} = \frac{x^3+2x^2-2x+1}{x(x-1)\left(x^2+x+1\right)} = -\frac{1}{x} + \frac{2}{3}\cdot\frac{1}{x-1} + \frac{4}{3}\cdot\frac{x+2}{x^2+x+1}$$

Zadatak 51(d)

Izračunajte integral $I := \int \frac{x^3 + 2x^2 - 2x + 1}{x^4 - x} \,\mathrm{d}x$.

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^3 + 2x^2 - 2x + 1}{x^4 - x} = \frac{x^3 + 2x^2 - 2x + 1}{x(x-1)\left(x^2 + x + 1\right)} = -\frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x-1} + \frac{4}{3} \cdot \frac{x+2}{x^2 + x + 1}$$

pa je

$$I = \int \left(-\frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x-1} + \frac{4}{3} \cdot \frac{x+2}{x^2 + x + 1} \right) \mathrm{d}x$$

Zadatak 51(d)

Izračunajte integral $I := \displaystyle\int \frac{x^3 + 2x^2 - 2x + 1}{x^4 - x}\,\mathrm{d}x$.

Rješenje. Rastavom na parcijalne razlomke (sami) dobivamo

$$\frac{x^3 + 2x^2 - 2x + 1}{x^4 - x} = \frac{x^3 + 2x^2 - 2x + 1}{x(x-1)\left(x^2 + x + 1\right)} = -\frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x-1} + \frac{4}{3} \cdot \frac{x+2}{x^2 + x + 1}$$

pa je

$$\begin{aligned} I &= \int \left(-\frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x-1} + \frac{4}{3} \cdot \frac{x+2}{x^2+x+1} \right) \mathrm{d}x \\ &= -\ln|x| + \frac{2}{3} \ln|x-1| + \frac{4}{3} \underbrace{\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x}_{=:I_1}. \end{aligned}$$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x$.

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x.$

Imamo $$I_1 = \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$
$$= \begin{bmatrix} t = x+\frac{1}{2} \;\Rightarrow\; x = t-\frac{1}{2} \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix}$$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x$.

Imamo

$$I_1 = \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

$$= \left[\begin{matrix} t = x + \frac{1}{2} \;\Rightarrow\; x = t - \frac{1}{2} \\ \mathrm{d}t = \mathrm{d}x \end{matrix}\right] = \int \frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}\,\mathrm{d}t$$

$$\int \mathrm{d}x = x + C$$

$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

$$\int e^x\,\mathrm{d}x = e^x + C$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x$.

Imamo

$$I_1 = \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

$$= \begin{bmatrix} t = x+\frac{1}{2} \;\Rightarrow\; x = t-\frac{1}{2} \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix} = \int \frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}\,\mathrm{d}t$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{t}{t^2+\frac{3}{4}}\,\mathrm{d}t}_{\substack{\text{supstitucijom } u=t^2+\frac{3}{4} \\ \text{ili pogodimo}}} + \frac{3}{2}\underbrace{\int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t}_{\text{tablični integral}}$$

$\int \mathrm{d}x = x + C$

$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$

$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$

$\int e^x\,\mathrm{d}x = e^x + C$

$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$

$\int \cos x\,\mathrm{d}x = \sin x + C$

$\int \sin x\,\mathrm{d}x = -\cos x + C$

$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$

$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$

$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$

$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$

$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x.$

Imamo

$$I_1 = \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

$$= \left[\begin{array}{c} t = x+\frac{1}{2} \;\Rightarrow\; x = t-\frac{1}{2} \\ \mathrm{d}t = \mathrm{d}x \end{array}\right] = \int \frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}\,\mathrm{d}t$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{t}{t^2+\frac{3}{4}}\,\mathrm{d}t}_{\substack{\text{supstitucijom } u=t^2+\frac{3}{4} \\ \text{ili pogodimo}}} + \frac{3}{2}\underbrace{\int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t}_{\text{tablični integral}}$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2}{\sqrt{3}}t + C_1$$

$$\int \mathrm{d}x = x + C$$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$
$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$
$$\int e^x\,\mathrm{d}x = e^x + C$$
$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \cos x\,\mathrm{d}x = \sin x + C$$
$$\int \sin x\,\mathrm{d}x = -\cos x + C$$
$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$
$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$
$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(d)

Preostaje izračunati integral $I_1 := \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x.$

Imamo

$$I_1 = \int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \overset{\text{Trik B}}{=} \int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

$$= \begin{bmatrix} t = x+\frac{1}{2} \;\Rightarrow\; x = t-\frac{1}{2} \\ \mathrm{d}t = \mathrm{d}x \end{bmatrix} = \int \frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}\,\mathrm{d}t$$

$$\overset{\text{Trik A}}{=} \underbrace{\int \frac{t}{t^2+\frac{3}{4}}\,\mathrm{d}t}_{\substack{\text{supstitucijom } u=t^2+\frac{3}{4} \\ \text{ili pogodimo}}} + \frac{3}{2}\underbrace{\int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t}_{\text{tablični integral}}$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2}{\sqrt{3}}t + C_1$$

$$= \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C_1.$$

$$\int \mathrm{d}x = x + C$$

$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

$$\int e^x\,\mathrm{d}x = e^x + C$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \quad (a > 0,\; a \neq 1)$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a > 0)$$

Zadatak 51(d)

Dakle,

$$I = -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{4}{3}\underbrace{\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x}_{=:I_1},$$

a upravo smo izračunali da je

$$I_1 = \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C_1.$$

Zadatak 51(d)

Dakle,

$$I = -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{4}{3}\underbrace{\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x}_{=:I_1},$$

a upravo smo izračunali da je

$$I_1 = \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C_1.$$

Prema tome,

$$I = -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{4}{3}\left(\frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\right) + C$$

Zadatak 51(d)

Dakle,

$$I = -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{4}{3}\underbrace{\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x}_{=:I_1},$$

a upravo smo izračunali da je

$$I_1 = \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C_1.$$

Prema tome,

$$\begin{aligned} I &= -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{4}{3}\left(\frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\right) + C \\ &= -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{2}{3}\ln\left(x^2+x+1\right) + \frac{4\sqrt{3}}{3}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$